PLANE FLOWS OF AN IDEAL GAS WITH INFINITE ELECTRICAL CONDUCTIVITY, IN A MAGNETIC FIELD NOT PARALLEL TO THE FLOW VELOCITY

(PLOSKIE TECHENIA IDEALNOGO GAZA S BESKONECHNOI ELEKTROPROVODNOST'IU V MAGNITNOM POLE, NE PARALLELNOM SKORISTI POTOKA)

PMM Vol.24, No.1, 1960, pp.100-110

M. N. KOGAN
(Moscow)

(Received 8 October 1959)

This paper presents an analysis and classification of flows. Hyperbolic and elliptic-hyperbolic types of flow are examined. It is shown that in flow past corners in the hyperbolic regime the turning of the flow occurs successively through two compression shocks or expansion waves. A method for calculating such flows is given. It is shown that elliptical-hyperbolic flows may be decomposed into an elliptical part which dies out at infinity and a hyperbolic part which does not. The character of flow past currents is investigated. It is shown that under the influence of a magnetic field component which is perpendicular to the flow, the perturbations induced by the currents are not shielded by the flow. In an ideal infinitely conducting fluid, in the presence of a small perpendicular field, a magnetic boundary layer develops around currents.

1. Equations and characteristics. As is well known [1], the equations of magnetohydrodynamics for an ideal gas with infinite electrical conductivity have the form

$$\operatorname{div}(\rho \mathrm{V}) = 0, \qquad (\mathrm{V}\nabla)\,\mathrm{V} = -\frac{\nabla p}{\rho} - \frac{1}{4\pi\rho} H \times \operatorname{rot} \mathrm{H}$$
$$\operatorname{div} \mathrm{H} = 0, \qquad \operatorname{rot}(\mathrm{V}\times \mathrm{H}) = 0, \qquad \mathrm{V}\operatorname{grad}\left(\frac{p}{\rho^{\varkappa}}\right) = 0 \tag{1.1}$$

Here ρ is the density, p is the pressure, $\mathbf{V}$ is the velocity, and $\mathbf{H}$ is the magnetic field.

In order to obtain more simple and descriptive results we restrict ourselves to linear theory. After linearization, the system of equations (1.1) takes the form

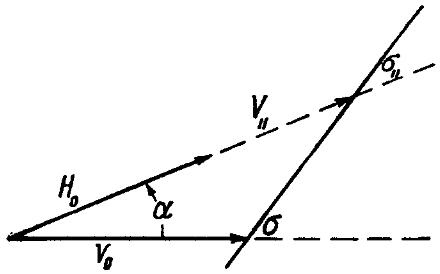

FIG. 1.

$$
\begin{gathered}
\rho_0\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+V_0\frac{\partial \rho}{\partial x}=0 \\
\frac{\partial H_x}{\partial x}+\frac{\partial H_y}{\partial y}=0 \qquad (1.2)\\
V_0\frac{\partial u}{\partial x}=-\frac{1}{p}\frac{\partial p}{\partial x}+\frac{H_{y_0}}{4\pi\rho_0}\left(\frac{\partial H_x}{\partial y}-\frac{\partial H_y}{\partial x}\right)\\
V_0\frac{\partial v}{\partial x}=-\frac{1}{\rho_0}\frac{\partial p}{\partial y}+\frac{H_{x_0}}{4\pi\rho_0}\left(\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}\right)\\
V_0H_y+H_{y_0}u-vH_{x_0}=0, \qquad p=\frac{\varkappa p_0}{\rho_0}\rho
\end{gathered}
$$

where the symbols without indices denote small perturbations on the values of the corresponding quantities at infinity, denoted by symbols with index 0.

The inclination of the characteristics of Equations (1.1) to the velocity vector is determined by the equations

$$
\begin{gathered}
y'^4[(M^2-N_x^2)(1-M^2)+M^2N_y^2]+2y'^3N_xN_y+\\
+y'^2[M^2-N_x^2(1-M^2)-N_y^2(1-M^2)]+2N_xN_yy'-N_y^2=0\,,(1.3)\\
\left(N_x=\sqrt{\frac{H_x^2}{4\pi\varkappa p}},\ N_y=\sqrt{\frac{H_y^2}{4\pi\varkappa p}}\right)
\end{gathered}
$$

Here y' is the tangent of the angle of inclination of a characteristic to the velocity vector, M is the Mach number. This equation may be derived, as is done in the theory of differential equations, by setting up the kinematic and dynamic conditions of compatibility [2]. But it may be obtained more simply if, in the equation which determines the velocity of propagation of magnetohydrodynamic waves (cf., for instance, Equation 52.12 in [1]), the latter is equated to the velocity component which is normal to the wave front. In view of the complexity of Equation (1.3) it is difficult to see the behavior of its roots. A more descriptive presentation of the characteristics of Equations (1.1) may be obtained if they are considered as shock waves of vanishing strength. Inasmuch as the inclination of shock waves of vanishing strength is known [3] for the case where the vectors **H** and **V** are parallel, the inclination of these waves for arbitrary directions of **H** and **V** may be obtained by choosing a corresponding moving system of coordinates. From Fig. 1 it may be seen that the solutions of Equation (1.3) may be presented in the following parametric form:

$$\operatorname{tg}\sigma = y' = \frac{\operatorname{tg}\sigma_{||} + \operatorname{tg}\alpha}{1 - \operatorname{tg}\sigma_{||}\operatorname{tg}\alpha}, \qquad M = \frac{M_{||}}{\cos\alpha}\,\frac{\operatorname{tg}\sigma_{||}}{\operatorname{tg}\sigma_{||} + \operatorname{tg}\alpha} \tag{1.4}$$

where

$$\operatorname{tg}^2\sigma_{||} = \frac{M_{||}^2 - N^2(1 - M_{||}^2)}{(M_{||}^2 - 1)(M_{||}^2 - N^2)}, \qquad N^2 = \frac{H^2}{4\pi\varkappa p}$$

Given the values of $M_{||}$ in those regions in which $\operatorname{tg}\sigma_{||}$ exists, it is easy to trace the behavior of the characteristics. Figs. 2 and 3 show the variation of M and $\operatorname{tg}\sigma$ with $M_{||}$ corresponding to large and small

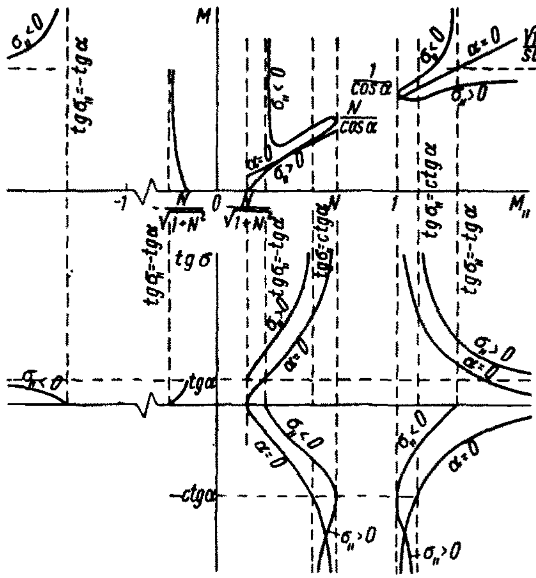

FIG. 2.

values of $\operatorname{tg}\alpha$, for $N < 1$. For $N > 1$ an analogous picture is obtained. For small values of α there are two hyperbolic and two elliptic-hyperbolic regimes; for large α there is one elliptic-hyperbolic (for small M) and one hyperbolic region (Fig. 4). In the elliptic-hyperbolic regimes there are two characteristics, and in the hyperbolic ones there are four. As in [3] we shall call the first hyperbolic region quasi-hyperbolic, and the second one fully hyperbolic, or simply hyperbolic.

Along the characteristics the following conditions are satisfied:

$$(y'N_yN_x + M^2N_y^2y'^2 - N_y^2)\frac{u'}{V_0} - (y'^2N_x^2 + M^2N_xN_yy'^3 - y'N_xN_y)\frac{H_x'}{H_{x_0}} -$$
$$- [y'^3(M^2 - N_x^2) + y'^2N_xN_y(1 + M^2)]\frac{v'}{V_0} -$$

$$-[y'^2 + y'^3 N_x N_y + y' N_x N_y] \frac{p'}{\varkappa p_0} = 0 \qquad (1.5)$$

Here the primes denote differentiation with respect to x, along a characteristic.

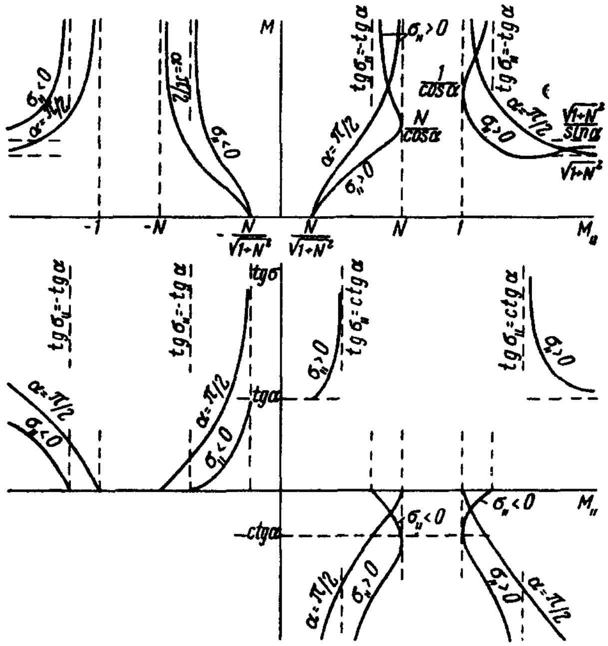

FIG. 3.

2. Flow in the hyperbolic region. In [3] flows with $\mathbf{H}_0 \parallel \mathbf{V}_0$ are analyzed. Let us investigate the other limiting case, where $\mathbf{H}_0 \perp \mathbf{V}_0$. Expression (1.5) takes the form

$$N^2 (M^2 y'^2 - 1) \frac{u'}{V_0} + y' N^2 (1 - M^2 y'^2) \frac{H_x'}{H_0} - y'^3 M^2 \frac{v'}{V_0} - \frac{p'}{\varkappa p_0} y'^2 = 0 \qquad (2.1)$$

For this case, Equation (1.3) has the roots

$$y'^2_{1,2} = \frac{-[M^2 - N^2(1 - M^2)] \pm \sqrt{[M^2 - N^2(1 - M^2)]^2 + 4N^2 M^2 [(1 - M^2) + N^2]}}{2[(1 - M^2) + N^2] M^2} \qquad (2.2)$$

Here $N = N_{y_0}$, since $H_{x_0} = 0$, and the index 1 corresponds to the smaller root, index 2 to the larger one.

For $H \parallel V_0$ it was possible to speak, just as in ordinary hydrodynamics, about the flow about a body of given form, independently of the field inside the body. This is explained by the fact that the field at the wall of the body is always parallel to the wall and is independent of the field inside the body. The field inside the body simply determines the jump in field at the surface. If $V_0 \neq H_0$ then $uH_y - vH_x = V_0 H_{y0} \neq 0$, throughout the whole flow field.

It follows that on the surface of the body in the flow, $u_t H_n = V_0 H_{y0} \neq 0$, i.e. the normal component of the field is different from zero. In this case the field at the wall cannot have a discontinuity since then a surface current would develop, on which there would be a force tangent to the body, which is impossible in an ideal fluid. Therefore, it is not possible to investigate the problem of flow over a body of given form independently of the development of the field inside the body.

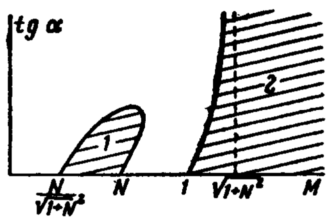

FIG. 4.

Let us investigate, for example, the flow about a thin profile at zero angle of attack. Inside the body there are no sources of magnetic field. Therefore, during the passage of the body the field can change by an amount which is of the order of the square of the body thickness. It follows that, to the accuracy being considered, $H_{x+} = H_{x-}$ and $H_{y+} = H_{y-}$ (the indices plus and minus refer, respectively, to upper and lower surfaces of the profile). From this condition and the second last equation of (1.2) it follows that $u_+ = u_-$. In addition, the profile shape is given, that is, $V_+ = - V_- = f(x)$.

For $a = 1/2\pi$ the hyperbolic type of flow occurs for values of $M > \sqrt{1 + N^2}$. In this case, disturbances cannot penetrate upstream. Therefore, Equations (2.1) may be integrated along the characteristics which go upstream, since in front of the body all disturbances are zero. This gives us four relations, for six unknowns H_{x+}, u_+, and p_+.

With the conditions $H_{x+} = H_{x-}$ and $u_+ = u_-$ there are enough equations to determine u_+, p_+ and H_{x+}. Then H_y is found from the second last equation of (1.2).

Solving this system for the problem under consideration, we obtain

$$\frac{u_+}{V_0} = y_1'^2 y_2'^2 \frac{M^2}{\Delta_2}(|y_1'| - |y_2'|)\frac{v_+}{V_0}, \qquad H_{x+} = H_{y-} = 0$$
$$\frac{p_+}{p_0} = \varkappa M^2 \frac{\Delta_3}{\Delta_2}\frac{v_+}{V_0}, \qquad p_- = p_+ \tag{2.3}$$
$$\Delta_k = |y_2'|^k (1 - M^2 y_1'^2) - |y_1'|^k (1 - M^2 y_2'^2)$$

From (1.4), when $\operatorname{tg}\alpha \to \infty$ we have $y' = -\operatorname{ctg}_{\|}\sigma_{11}$ and $M = M_{\|}\operatorname{tg}\sigma_{\|}$. From (2.2) and Fig. 3 it is easy to see that the small root y_1 corresponds to values of $M_{\|}$ in the interval $N/\sqrt{1+N^2} \leqslant M_{\|} \leqslant N < 1$ for $N < 1$, and in the interval $N/\sqrt{1+N^2} \leqslant M_{\|} \leqslant 1$ for $N > 1$. The larger root y_2' corresponds to the value $M_{\|} > 1$ for $N < 1$, and $M_{\|} > N$ for $N > 1$. Therefore, $y_1'^2 M^2 < 1$ and $y_2'^2 M^2 > 1$. Consequently, $\Delta_k > 0$.

Thus, for $v_+ > 0$ the pressure on the wall $p_+ > 0$ and the velocity $u_+ < 0$, as in ordinary gas dynamics. However, where in ordinary gas dynamics the pressure increase occurs through a single compression shock, e.g. on a wedge, here it occurs through two successive shocks. In fact, let us examine any point between the characteristics issuing downstream from the vertex of the wedge (Fig. 5a). It is evident that through such a point there pass three characteristics from upstream infinity and one (with inclination $+ y_2'$) from the body surface. Integrating (2.1) along these characteristics we will obtain four relations for determining four quantities u, v, p and H_x at the point in question.

Solving these equations we find

$$\frac{v}{v_+} = \frac{y_2'^2(1 - M^2 y_1'^2)}{\Delta_2}, \qquad \frac{p}{p_+} = \frac{|y'_2|^3(1 - M^2 y_1^2)}{\Delta_3}, \qquad \frac{u}{u_+} = \frac{|y'_2|}{|y_2'| - |y_1'|}$$

From this it may be seen that for $v_+ > 0$, $v > 0$ and $p > 0$. Also, $v/v_+ < 1$ and $p/p_+ < 1$. Thus, the flow is compressed first by the first shock and then by the second. For $N \to 0$ the angle of inclination and strength of the second shock tend toward zero, while the first shock becomes the shock of ordinary gas dynamics.

We note that at the first shock the flow is slowed down ($u < 0$) while at the second one it is accelerated ($u/u_+ > 1$). In fact, to the shock with inclination y_2' there corresponds a value of $M_{\|} > 1$, and to the shock with inclination y_1' a value of $M_{\|}$ between $N/\sqrt{1+N^2}$ and $N < 1$. In the first case, as shown in [3] the end of the velocity vector downstream of a shock with inclination $\sigma_{\|}$, corresponding to σ_2, appears in the first quadrant, while in the second case it appears in the second quadrant (Fig. 6). In passing through a real shock with inclination $\sigma_{1,2}$, the end of the velocity vector appears, correspondingly in quadrants 1' and 2'. In the first case the velocity decreases, in the second case increases.

We shall also investigate the flow over a flat plate at angle of attack (Fig. 5b). In this case, $v_+ = v_-$. Again, proceeding along characteristics from upstream infinity we obtain

$$u_+ = u_- = 0, \qquad p_+ = -p_-, \qquad H_{y+} = H_{y-} = 0$$

$$\frac{p_+}{p_0} = \frac{\varkappa \Delta_2 M^2 v_+}{\Delta_1 V_0}, \qquad \frac{H_{x+}}{H_0} = \frac{N^2 M^2 |y_1'| \, |y_2'| \, (|y_2'| - |y_1'|)}{\Delta_2} \frac{v_+}{V_0} \qquad (2.4)$$

Analogously to the foregoing, we have at a point between the characteristics,

$$\frac{v}{v_+} = \frac{|y_2'| (1 - M^2 y_1'^2)}{\Delta_1}, \quad \frac{p}{p_+} = \frac{y_2'^2 (1 - M^2 y_1'^2)}{\Delta_2}, \quad \frac{u}{v_+} = - \frac{M^2 y_1'^2 y_2'^2}{N^2 \Delta_1}$$

Thus, in this case also, for $v_+ > 0$ the compression occurs through two shocks. But we note that for a given flow deflection in the two cases we have considered the shock systems will be different. It is evident that for $v_+ > 0$ the flow deflection occurs by means of expansion waves. For $N \to 0$ the second shock (or expansion wave) lies along the body.

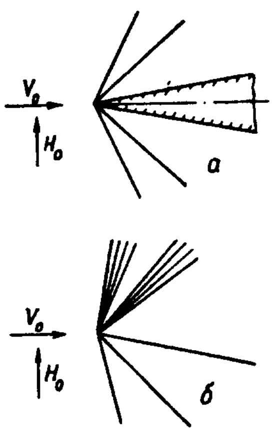

FIG. 5.

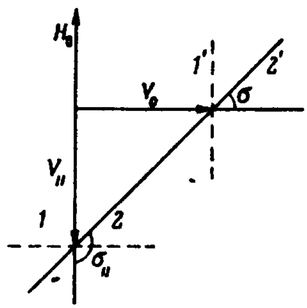

FIG. 6.

3. Flow over currents. Up to now it was assumed that there were no currents in the body. We shall now investigate the flow around a body containing currents perpendicular to the plane of the flow.

For a simple example we shall investigate a flat plate at zero angle of attack (the conducting layer is assumed to be isolated from the flow). Let

$$H_{x_0} = 0, \qquad H_{y0} = H_0$$

In this case the following conditions must be fulfilled on the surface of the body:

$$H_{y+}=H_{y-}, \qquad v_+=v_-=0, \qquad u_+=u_-, \qquad H_{x+}-H_{x-}=f(x)$$

where the function $f(x)$ is determined by the distribution of current along the plate. Proceeding along the characteristics from upstream infinity, as in Section 2, we find

$$H_+=-H_-=\tfrac{1}{2}f(x), \qquad \frac{u_+}{V_0}=-|y_1'|\cdot|y_2'|\frac{\Delta_1}{\Delta_2}\frac{H_{x+}}{H_0}$$
$$p_+=p_-=-\frac{\varkappa p_0 N^2(1-M^2y_1'^2)(1-M^2y_2'^2)(|y_2'|-|y_1'|)}{\Delta_2}\frac{H_{x+}}{H_0} \tag{3.1}$$

We assume for simplicity that $f(x)$ = const and investigate a point between the characteristics which proceed downstream from the nose of the plate (Fig. 7). Proceeding to this point along the three characteristics from infinity and along one characteristic from the body, we obtain

$$\frac{u}{u_+}=-\frac{|y_1'|(1-M^2y_2'^2)}{\Delta_1}, \qquad \frac{p}{p_+}=\frac{|y_2'|}{|y_2'|-|y_1'|}$$
$$\frac{H_+}{H_{x+}}=-\frac{y_1'^2(1-M^2y_2'^2)}{\Delta_2}$$

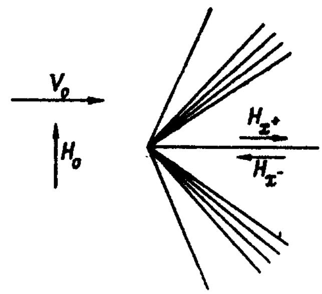

FIG. 7.

For $H_{x+}>0$ we have $p_+>0$, $u_+<0$, and in addition, $p>0$ and $p/p_+>1$.

Therefore, the flow first goes through a shock in which the gas is strongly compressed, after which it is expanded a certain amount by an expansion wave. For $N\to 0$ this expansion wave lies along the body, going over into a tangential discontinuity of magnetic field, and in the region between the shock and the expansion wave the disturbance approaches zero. Thus, the flow fully screens the currents only in the absence of a transverse magnetic field. In all other cases, disturbances created by the currents penetrate into the flow.

Evidently, for $H_{x+}<0$ there is first an expansion wave, followed by a shock wave.

For large but finite conductivity, the shock (or expansion wave) becomes a certain layer which, for $N\to 0$, approaches the body to become a magnetic boundary layer of the type investigated in [4].

For $M<\sqrt{1+N^2}$ and $H_{y0}\neq 0$ disturbances penetrate into all flows. However, for small values of H_{y0}, in an ideal infinitely conducting fluid, there appears around the plate a magnetic boundary layer, in which the magnetic field changes from a value at the wall (determined by the flow) to practically zero (Fig. 8). This layer is similar to the one investigated in [5]. In order to clarify in the simplest possible manner the phenomena obtained here, we shall investigate the flow over a plate with current, in an infinitely conducting incompressible fluid (Fig. 8).* The thickness of the layer is proportional to H_{y_0}. Taking u and H_x to be

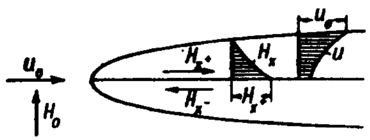

FIG. 8.

quantities of order unity, and making the usual assumptions of boundary layer theory, we find from one of the equations of motion that across the layer

$$p + \frac{H_x^2}{8\pi} = \text{const}$$

The other equation of motion becomes

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{4\pi\rho}\left(H_x\frac{\partial H_x}{\partial x} + H_y\frac{\partial H_x}{\partial y}\right) \tag{3.2}$$

In addition, the following equations must be satisfied:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} = 0, \qquad uH_y - vH_x = H_{y_0}u_0 \tag{3.3}$$

The first two equations of (3.3) make it possible to introduce the functions ψ and χ by means of the relations

$$u = \partial\psi/\partial y, \quad v = -\partial\psi/\partial x, \quad H_x = \partial\chi/\partial y, \quad H_y = -\partial\chi/\partial x \tag{3.4}$$

Differentiating the last equation of (3.3) and making use of the other two equations of (3.3) we obtain

$$u\frac{\partial H_y}{\partial x} + v\frac{\partial H_y}{\partial y} = H_x\frac{\partial v}{\partial x} + H_y\frac{\partial v}{\partial y}, \quad H_x\frac{\partial u}{\partial x} + H_y\frac{\partial u}{\partial y} = u\frac{\partial H_x}{\partial x} + v\frac{\partial H_x}{\partial y} \tag{3.5}$$

In Equations (3.2) and (3.5) it is convenient to change to the variables ψ and χ. We have

---

* Here we do not assume the disturbance to be small.

$$\frac{1}{4\pi\rho}\frac{\partial H_x}{\partial\psi}+\frac{\partial u}{\partial\chi}=0, \qquad \frac{\partial H_x}{\partial\chi}+\frac{\partial u}{\partial\psi}=0 \tag{3.6}$$

$$\frac{\partial H_y}{\partial\chi}+\frac{\partial v}{\partial\psi}=0 \tag{3.7}$$

The system (3.6) evidently has two families of characteristics $\psi\sqrt{4\pi\rho}+\chi=$ const. Along these characteristics the relations

$$\sqrt{4\pi\rho}u'\pm H_x'=0 \tag{3.8}$$

are satisfied. The characteristics corresponding to the upper sign we shall call characteristics of the first family, and those with the lower sign the second family.

Equation (3.7) makes possible a function $\Phi(\chi, \psi)$ such that

$$v=\partial\Phi/\partial\chi, \; H_y=-\partial\Phi/\partial\psi$$

Therefore, the last of equations (3.3) may be put in the form

$$u\frac{\partial\Phi}{\partial\psi}+H_x\frac{\partial\Phi}{\partial\chi}=-H_{y_0}u_0 \tag{3.9}$$

The variables ψ and χ are connected with the variables x and y by the relations

$$dx=\frac{1}{H_{y_0}u_0}(H_x d\psi-u d\chi), \qquad dy=\frac{1}{H_{y_0}u_0}(H_y d\psi-v d\chi) \tag{3 10}$$

As an example, let us investigate the flow over a flat plate with a current distributed according to the relation $H_{x+}=kx=f(\chi)$.

In the χ, ψ plane the problem is formulated as follows:

$$H_{x+}=kx, \; v=0 \;\text{ for } \psi=0$$

At infinity, along characteristics of the first family,

$$H_x=0, \; u=u_0 \;\text{ for } \psi\to\infty$$

Along characteristics of the first family we have

$$\sqrt{4\pi\rho}\,u+H_x=u_0\sqrt{4\pi\rho} \tag{3.11}$$

It follows that on the body $u(\chi, 0)=u_0-H_{x+}/\sqrt{4\pi\rho}$. Then, from (3.10), we have

$$d\chi=\frac{H_{y_0}u_0\sqrt{4\pi\rho}}{kx-u_0\sqrt{4\pi\rho}}dx, \quad \text{or} \quad x=\frac{u_0\sqrt{4\pi\rho}}{k}(1-e^{A\chi}) \quad \left(A=\frac{k}{H_{y_0}u_0\sqrt{4\pi\rho}}\right)$$

It is easy to see that u and H_x are constant along characteristics of the second family $\lambda = \sqrt{4\pi\rho}\ \psi + \chi =$ const for $\lambda < 0$, and that $u = u_0$ and $H_x = 0$ for $\lambda > 0$.

Thus, for $\lambda < 0$,

$$u = u_0 e^{A\lambda}, \qquad H_x = u_0\sqrt{4\pi\rho}\,(1 - e^{A\lambda}) \tag{3.12}$$

Putting these expressions in Expression (3.9) and integrating, we obtain

$$\Phi = \frac{H_{y_0}}{A\sqrt{4\pi\rho}} \ln\left(1 - \frac{A\sqrt{4\pi\rho}}{e^{A\lambda}}\,\psi\right) \tag{3.13}$$

and

$$H_y = H_{y_0}\,\frac{e^{-A\lambda}(1 - A\sqrt{4\pi\rho}\,\psi)}{1 - A\sqrt{4\pi\rho}\,\psi e^{-A\lambda}}, \quad v = H_{y_0}\,\frac{A\psi e^{A\lambda}}{1 - \sqrt{4\pi\rho}\,\psi e^{-A\lambda}}$$

Putting (3.14) into (3.10) with $\lambda = 0$ we find a line which is the edge of the boundary layer in the physical plane:

$$y = -\frac{H_{y_0}}{K}\ln\left(1 - \frac{kx}{u_0\sqrt{4\pi\rho}}\right)$$

For $kx \to u_0\sqrt{4\pi\rho}$ we have $y \to \infty$. This point corresponds to a separation of the flow, $[u/u_0\sqrt{4\pi\rho}/k,\ 0) = 0]$.

We note that the force acting on the current flowing in the body, in the presence of the flow of an infinitely conducting fluid or gas, is different from the force which acts on the same current in the absence of flow. The resistance created by a transverse magnetic field is similar, in the well-known sense, to a resistance which is dependent on viscosity.

4. Flows in the elliptic-hyperbolic regime. According to the classification given in Section 1, we call those flows elliptic-hyperbolic in which only two characteristics appear. We shall investigate this flow in detail for $a = 1/2\,\pi$. From Fig. 3 and Equation (2.2) it follows that at every point of the flow there are two families of characteristics corresponding to $\sigma > 0$ and $\sigma < 0$. Along the characteristics relations (2.1) are satisfied.

We assume that (just as in the usual flows of elliptic type) all perturbations die out at infinity. Then two relations (2.1), satisfied along the characteristics, allow p to be expressed in terms of u and H_x in terms of v. Solving this system for given boundary conditions on the velocity, we at the same time determine completely the distribution of magnetic field, in particular, at the boundary of the body. It is clear that values so obtained for the field at the boundary of the body can be continuously joined with the magnetic field inside the body only for a special choice of the boundary conditions on the velocities and the currents inside the body. In the general case, the condition for continuity of the field at the body boundary and the condition for the dying out of the perturbations at infinity are not compatible. From this it follows that, within the framework of the linear theory being considered by us, the perturbations at infinity do not die out. In a nonlinear approximation, naturally, all perturbations die out at infinity. Therefore, the fact that perturbations do not die out along characteristics in the linear theory indicates that in the real flow the characteristics must terminate somewhere in the flow.

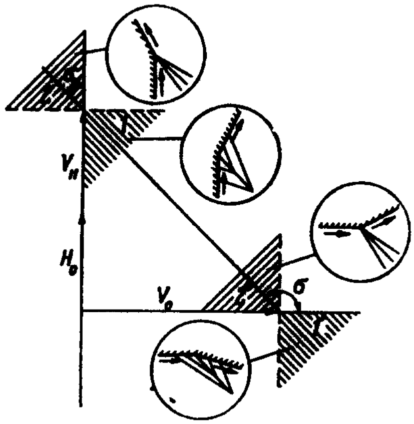

FIG. 9.

In the case under consideration they can run into shock waves. Indeed, since the characteristics are shock waves of vanishing strength, the existence of characteristics points to the existence of shock waves in the flow regime being investigated.

From Fig. 3 it was seen that shock waves exist for all values of M from zero to $m = \sqrt{1 + N^2}$.

Let us investigate which of two possible shock waves (directed upstream or downstream) can extend from the body. Here, just as in ordinary gas dynamics, it is necessary to distinguish a wave leaving the body from one coming into it. Let us assume that from some point on the upper surface of the body (Fig. 9) a shock or an expansion wave goes upstream at an inclination angle σ. From Fig. 9 it may be seen that the waves being considered correspond to values of the parameter $M_{\|}$ falling in the intervals

$$N/\sqrt{1+N^2} < M_{\|} < N \text{ for } N < 1 \text{ и } N/\sqrt{1+N^2} < M_{\|} < 1 \quad \text{for} \quad N > 1.$$

For these values of the parameter, as shown in [3] for a flow with $a = 0$, obtained from that examined above by the superposition of an appropriate velocity downstream of the wave, two regimes are possible as shown in Fig. 9. For that case, if the wave is a shock wave, the end of the velocity vector behind the shock will lie in Quadrant 1, if the wave is an expansion wave, it will lie in Quadrant 2. In the flow with $a = \pi/2$ the end of the velocity vector after the wave appears, correspondingly, in Quadrants 1′ and 2′. From the flow schemes shown in Fig. 9 it is seen that on the upper surface a wave directed upstream will be an incoming wave. On the other hand, on the lower surface, where the wave with the inclination under consideration is directed downstream, the wave is an outgoing one. Thus, in the flow being investigated only waves outgoing downstream are possible. It follows that only characteristics which go out from the body downstream can run into shock waves; along these characteristics perturbations do not die out in the linear approximation; along characteristics which go upstream the perturbations tend toward zero.

In accordance with the analysis given, the following relations are valid at an arbitrary point of the flow, according to Equation (2.1):

$$L_{\pm}(x; y) = F(\lambda), \qquad L_{\mp}(x, y) = 0 \qquad (\lambda = x \mp y / |y_1'|) \tag{4.1}$$

where

$$L_{\pm}(x; y) = N^2 (M^2 y_1'^2 - 1) \frac{u(x; y)}{V_0} \pm |y_1'| N^2 (1 - M^2 y_1'^2) \frac{H_x(x; y)}{H_0} \mp$$
$$\mp |y_1'|^3 M^2 \frac{v(x;y)}{V_0} - \frac{p(x;y)}{\varkappa p_0} y_1'^2$$

and $F(\lambda)$ is a certain, so far unknown, function of λ. The upper sign corresponds to positive y_1' and the lower sign to negative (the body is assumed to be near the axis $y = 0$). It is evident that $F(\lambda) = 0$ along characteristics which do not go through the body, since one end of such characteristics goes out to upstream infinity. With the help of (4.1) and the last two equations of (1.2) it is possible to eliminate from Equations (1.2) either H_x, H_y, p and ρ, or u, v, p and ρ. The result gives two systems of equations for u and v, and H_x and H_y respectively:

$$\left[1 - \frac{N^2 (1 - M^2 y_1'^2)}{y_1'^2}\right] \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{V_2 F'(\lambda)}{2 y_1'^2}$$

$$\frac{\partial u}{\partial y} - \frac{M^2 y_1'^2}{M^2 (1 - M^2 y_1'^2)} \frac{\partial v}{\partial x} = \frac{V_0 F'(\lambda)}{2 |y_1' N^2 (1 - M^2 y_1'^2)} \tag{4.2}$$

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} = 0$$

$$\frac{N^2 - [M^2 - N^2 (1 - M^2)] y_1'^2}{y_1'^2} \frac{\partial H_y}{\partial x} - N^2 \frac{\partial H_x}{\partial y} = \frac{H_0 F'(\lambda)}{2 y_1'^2} \tag{4.3}$$

It is easily verified that the operators on the left-hand sides of these equations are of elliptic type for $M < \sqrt{1 + N^2}$ and of hyperbolic type for $M > \sqrt{1 + N^2}$. The solution of the systems (4.2) and (4.3) may be represented in the form of a sum, the first terms of which (u_1, v_1, H_{x1}, H_{y1}) are functions of λ and satisfy a non-homogeneous system of ordinary differential equations, while the second terms (u_2, v_2, H_{x2}, H_{y2}) are functions of x and y and satisfy homogeneous systems of partial differential equations, each one of which reduces to Laplace's equation.

Integrating the system of ordinary differential equations we find

$$u_1 = K_1 F, \qquad v_1 = K_2 F \qquad \text{etc.} \tag{4.4}$$

The constants of integration are equal to zero, since on the characteristics which do not pass through the body the functions u_1, v_1, etc. must be equal to zero.

The functions (4.4) represent the hyperbolic part of the solution, which does not die out at infinity. The functions u_2, v_2, etc. are the elliptic part of the solution, which dies out at infinity. The possibility of separating the solution into a hyperbolic and an elliptic part justifies the naming of these flows elliptic-hyperbolic.

On the boundary of a body (a thin one, for simplicity) one has twelve unknown functions, $F_{\pm}$, $u_{2\pm}$, $v_{2\pm}$, $H_{x_{2\pm}}$, $H_{x_{1\pm}}$ and $p_{\pm}$. These functions are connected by the four relations (4.1), the two relations $u_{2\pm}/V_0 = -H_{y_{2\pm}}/H_0$, obtained from the second last equation of (1.2), and the four conditions $H_{x+} = H_{x-}$, $H_{y+} = H_{y-}$ and $v_+ = f_+(x)$ and $v_- = f_-(x)$. With the help of these relations it is possible to find two functions $G_{1,2}(H_{x_2}, H_{y_2}, u_2, v_2) = 0$, which make it possible to formulate the boundary value problem for the

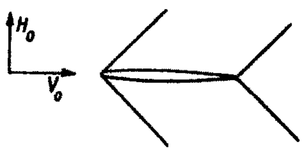

FIG. 10.

two Laplace equations obtained from (4.2) and (4.3). For flows with symmetry with respect to the axis $y = 0$ these functions are separable (i.e. $G_1(u_2, v_2) = 0$ and $G_2(H_{x_2}, H_{y_2}) = 0$), and the problem for each Laplace equation is solved separately.

A sketch of the shock waves which appear at the body, and in which the hyperbolic part of the solution has a discontinuity, is shown in Fig. 10.

BIBLIOGRAPHY

1. Landau, L.D. and Lifshitz, E.M., *Elektrodinamika sploshn'ikh sred (Electrodynamics of continuous media)*. GITTL, Moscow, 1957.

2. Zhigulev, V.N., Analiz slab'ikh razr'ivov v magnitnoi gidrodinamike (Analysis of weak discontinuities in magnetohydrodynamics). *PMM* Vol. 23, No. 1, 1959.

3. Kogan, M.N., Magnitodinamika ploskikh i osesimmetrichn'ikh techenii gaza c beskonechnoi elektrischeskoi provodomostiu (Magneto-dynamics of plane and axisymmetric flows of a gas with infinite electrical conductivity). *PMM* Vol. 23, No. 1, 1959.

4. Zhigulev, V.N., Teoriia magnitovo pogranichnovo sloia (Theory of the magnetic boundary layer). *Dokl. Akad. Nauk SSSR* Vol. 124, No. 5, 1959.

5. Kogan, M.N., O ploskom techenii bezkonechno provodiashchei zhidkosti s pochti parallel'nymi vektorami magnitnovo polia i skorosti (On the plane flow of an infinitely conducting fluid with nearly parallel velocity and magnetic field vectors). *Dokl. Akad. Nauk SSSR* Vol. 130, No. 2, 1960.

Translated by A.R.